

# On normal subgroups in the fundamental groups of complex surfaces

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## Abstract

We show that for each aspherical compact complex surface  $X$  whose fundamental group  $\pi$  fits into a short exact sequence

$$1 \rightarrow K \rightarrow \pi \rightarrow \pi_1(S) \rightarrow 1$$

where  $S$  is a compact hyperbolic Riemann surface and the group  $K$  is finitely-presentable, there is a complex structure on  $S$  and a nonsingular holomorphic fibration  $f : X \rightarrow S$  which induces the above short exact sequence. In particular, the fundamental groups of compact complex-hyperbolic surfaces cannot fit into the above short exact sequence. As an application we give the first example of a non-coherent uniform lattice in  $Isom(\mathbb{H}_{\mathbb{C}}^2)$ .

## 1 Introduction

The goal of this paper is threefold:

- (a) We will establish a restriction on the fundamental groups of compact aspherical complex surfaces.
- (b) We find the first examples of incoherent uniform lattices in  $PU(2, 1)$ .
- (c) We show that the answer to the Question 1 below is negative in the class of uniform lattices in  $PU(2, 1)$ .

**Question 1** *Is there a Gromov-hyperbolic group  $\pi$  which fits into a short exact sequence:*

$$1 \rightarrow K \rightarrow \pi \rightarrow Q \rightarrow 1$$

*where  $K$  and  $Q$  are closed hyperbolic surface groups?*

Suppose that  $X$  is an aspherical compact complex surface whose fundamental group  $\pi$  fits into a short exact sequence

$$1 \rightarrow K \rightarrow \pi \rightarrow Q = \pi_1(S) \rightarrow 1$$

where  $S$  is a compact hyperbolic Riemann surface and the group  $K$  is finitely-presentable. The main theorem of this paper is

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**Theorem 2** *Under the above assumptions there a complex structure on  $S$  and a nonsingular holomorphic fibration  $f : X \rightarrow S$  which induces the above short exact sequence.*

**Remark 3** *Actually in Theorem 2 it is enough to assume that  $Q$  is a torsion-free group with nonzero  $\beta_1^{(2)}(Q)$ , the 1-rst  $L_2$ -Betti number. On the other hand, in this case we have to assume that  $X$  is Kähler. Our proof also works under the assumption that the group  $K$  is of the type  $FP_2$ .*

After proving Theorem 2 I have learned that J. Hillman [10] proved the same result under stronger assumption that  $K$  is the fundamental group of a compact Riemann surface. Our methods seem to be completely different except application of the result of [1]. Later it turned out that the same result as Hillman's was independently proven by D. Kotschick.

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## 2 Milnor fibration

Let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a nonconstant holomorphic function, we assume that  $0 \in \mathbb{C}^2$  is a critical point of  $f$ . Let  $S_\epsilon = S_\epsilon(0)$  be a sufficiently small metric sphere in  $\mathbb{C}^2$  centered at the origin. Let  $B_\epsilon(0)$  denote the closed  $\epsilon$ -ball centered at the origin. Let  $K := f^{-1}(0) \cap S_\epsilon$ , this is a smooth knot (or link) in the 3-sphere. The *Milnor fibration*  $\phi : S_\epsilon - K \rightarrow S^1$  associated with  $f$  is defined as  $\phi(z, w) = f(z, w)/|f(z, w)|$ , see [16, §4].

Below we list some properties of  $\phi$  (see [16, §4], [6]):

- (a) If  $\epsilon$  is sufficiently small then  $\phi$  determines a smooth fibration of  $S_\epsilon - K$  over  $S^1$ .
- (b) Fibers of  $\phi$  are connected provided that the germ of  $f$  at zero is reduced, otherwise  $\phi$  will have disconnected fibers.
- (b) The knot (link)  $K$  is distinct from a single unknot in  $S^3$  unless the germ of  $f$  at 0 is isomorphic to  $((z, w) \mapsto z^p, (0, 0))$ .
- (c) If  $K$  not an unknot, then each component of  $\phi^{-1}(t), t \in S^1$ , is not simply-connected.
- (d) Let  $r > 0$  be sufficiently small. Consider  $s \in C_r(0)$ , a point on the unit circle in  $\mathbb{C}$  centered at zero. Let  $\mathcal{F}_{\epsilon,s} := f^{-1}(s) \cap B_\epsilon$ . The two surfaces  $\mathcal{F}_{\epsilon,s}$  and  $F_{\epsilon,s} = \phi^{-1}(s/|s|) - f^{-1}(B_r(0))$  share common boundary. There exists an isotopy of  $\mathcal{F}_{\epsilon,r}$  to  $F_{\epsilon,r}$  within  $B_\epsilon(0)$  which is the identity on the boundary of each surface.

## 3 Multicurves

**Definition 1** *Let  $f : X \rightarrow S$  be a nonconstant proper holomorphic map from a connected complex surface  $X$  to a Riemann surface (i.e. complex curve)  $S$ . We will say that  $f$  is a **nonsingular holomorphic fibration** if  $f$  is a submersion.*

Clearly the mapping  $f$  as above is a real-analytic fibration, however in most cases it does not determine a locally trivial holomorphic bundle. If  $f$  is not a submersion we will still think of it as a **singular** fibration, we shall use the notation  $\mathcal{F}_t$  to denote the fiber  $f^{-1}(t)$  of  $f$  over  $t \in S$ .

**Definition 2** Let  $f : X \rightarrow D^2$  be a nonconstant proper holomorphic map with connected fibers where  $X$  is a 2-dimensional complex surface and  $D^2$  is the unit disk in  $\mathbb{C}$ . We assume that the origin is the only critical value of  $f$ . The singular fiber  $C = f^{-1}(0)$  is called a **multicurve** if it is a smooth curve of the multiplicity  $> 1$ . In other words, the germ of  $f$  at each point  $c \in C$  is equivalent to the map  $(z, w) \mapsto z^n, n > 0, z, w \in \mathbb{C}$ . The number  $n$  is the multiplicity of  $C$ .

Let  $t \in D^2 - 0$ . Define the maps

$$\iota_* : H_2(f^{-1}(t)) \rightarrow H_2(X) \cong H_2(C)$$

$$\iota_\# : \pi_1(f^{-1}(t)) \rightarrow \pi_1(X) \cong \pi_1(C)$$

induced by the inclusion  $\iota : f^{-1}(t) \hookrightarrow X$ .

**Lemma 4** If  $C$  is a multicurve then the map  $\iota_*$  is not surjective. Assume that  $C$  is a non-simply-connected multicurve. Then the map  $\iota_\#$  is not onto.

*Proof.* Consider  $Y = f^{-1}(D) \subset X$  where  $D = \{z \in \mathbb{C} : |z| \leq |t|\}$  is the closed disk in  $D^2$  containing  $t$ . The inclusion  $Y \hookrightarrow X$  is a homotopy-equivalence so we restrict our discussion to  $Y$ . The map  $C \hookrightarrow Y$  is a homotopy-equivalence, thus the fundamental class of  $C$  generates  $H_2(Y)$ . The dual generator of  $H_2(Y, \partial Y)$  is represented by 2-disk  $\Delta \subset Y$  which is transversal to the fibers of  $f$  and  $\partial\Delta \subset \partial Y$ . Since  $C$  is a multicurve, the algebraic intersection number  $[f^{-1}(t)] \cdot [\Delta] = n > 1$ , where  $n$  is the multiplicity of  $C$ . Thus  $[f^{-1}(t)] = n[C]$  which proves the first assertion.

The map  $\iota_\#$  is injective (since  $\iota$  is homotopic to a covering  $f^{-1}(t) \rightarrow C$ ). Thus  $n = |\pi_1(C) : \iota_\#(\pi_1(f^{-1}(t)))|$ , this proves the second assertion.  $\square$

## 4 Proof of the main theorem

If  $\pi_1(X)$  fits into short exact sequence

$$1 \rightarrow K \rightarrow \pi \rightarrow Q = \pi_1(S) \rightarrow 1$$

where  $S$  is a hyperbolic Riemann surface then it follows from Kodaira's classification theorem that  $X$  is a complex-algebraic surface. If  $X$  is assumed to be Kähler,  $Q$  torsion-free and  $\beta_1^{(2)}(Q) \neq 0$ , then  $Q$  is the fundamental group of a hyperbolic Riemann surface, moreover if  $\tilde{X}$  is the covering of  $X$  corresponding to  $K$  then there is a discrete faithful conformal action of  $Q$  on  $\mathbb{H}^2$  and a  $Q$ -equivariant proper holomorphic map

$$\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^2$$

with connected fibers (see [1]). In particular, the projection  $\pi_1(X) \rightarrow Q$  is induced by a holomorphic map  $f : X \rightarrow S$ , for the complex structure on  $S$  given by  $\mathbb{H}^2/Q$ .

The  $i$ -th  $L_2$ -Betti number  $\beta_i^{(2)}(G)$  of a finitely presentable group  $G$  is the dimension of the  $i$ -th reduced  $L_2$ -cohomology group  $\bar{l}_2 H^i(G)$ , we refer the reader to [9, Chapter 8] and [1] for the precise definitions. For our purposes it is enough to know that  $\beta_i^{(2)}(Q) > 0$  for each 2-dimensional finitely presentable group  $Q$  provided that  $\chi(Q) < 0$  (see [9, Chapter 8]). In particular, if  $Q$  is the fundamental group of a hyperbolic Riemann surface of finite type then  $\beta_1^{(2)}(Q) > 0$ . Thus, in any case we have a holomorphic map  $f : X \rightarrow S$ .

We start the proof with the simple case when  $f$  is a *holomorphic Morse function*, i.e. the germ of  $f$  at each critical point is equivalent to  $(z, w) \mapsto zw$ . The proof in this case is easier and it illustrates the idea of the proof in the general case.

Let  $d$  denote the hyperbolic metric on the unit disk in  $\mathbb{C}$ . We will suppose that the origin 0 is a regular value of  $\tilde{f}$ . Direct computations show that the function

$$\gamma : x \mapsto d(0, \tilde{f}(x))$$

is a real Morse function on  $\tilde{X}$  away from  $\tilde{f}^{-1}(0)$  and the Morse index of  $\gamma$  at each critical point in  $\tilde{X} - \tilde{f}^{-1}(0)$  is two. It is clear that  $r \in \mathbb{R}_+$  is a critical value of  $\gamma$  if and only if there is a critical value  $z \in \mathbb{H}^2$  of  $\tilde{f}$  within the distance  $r$  from the origin. Let  $\mathcal{F}$  denote the generic fiber of  $\tilde{f}$ . Thus the space  $\tilde{X}$  is obtained by attaching 2-handles to  $\mathcal{F} \times D^2$ . Each singular fiber of  $\tilde{f}$  is obtained from  $\mathcal{F}$  by “pinching” a certain collection of disjoint simple loops. Since  $\tilde{X}$  is aspherical, each of these loops is homotopically nontrivial and no two such loops are homotopic to each other. (Otherwise  $\tilde{X}$  contains a rational curve which then lifts to a homologically nontrivial 2-cycle in the universal cover of  $X$ .)

We now claim that the group  $\pi_1(\tilde{X})$  is finitely generated but not finitely presentable. Our proof follows an argument of Bestvina and Brady [3]. Since  $\tilde{X}$  is obtained from  $\mathcal{F} \times D^2$  by attaching only 2-handles, the fundamental group of  $\tilde{X}$  is the quotient of  $\pi_1(\mathcal{F})$ . Recall that  $\pi_1(\tilde{X})$  is finitely presentable, the epimorphism

$$\pi_1(\mathcal{F}) \rightarrow \pi_1(\tilde{X})$$

determines a finite generating set for  $\pi_1(\tilde{X})$  (i.e. the generators of  $\pi_1(\mathcal{F})$ ).

**Lemma 5** *Let  $G$  be a finitely presentable group and  $\{y_1, \dots, y_m\}$  be a finite generating set for  $G$ . Then there is a finite number of relators  $R_1, \dots, R_k$  such that  $\langle y_1, \dots, y_m | R_1, \dots, R_k \rangle$  is a presentation of  $G$ .*

*Proof.* Let  $\langle x_1, \dots, x_s | Q_1, \dots, Q_n \rangle$  be a finite presentation of  $G$ . There is a finite sequence of *Tietze transformations* (see for instance [14, §1.5]) which transform the generating set  $X = \{x_1, \dots, x_s\}$  to  $Y = \{y_1, \dots, y_m\}$ , simultaneously they transform system of relators  $Q_1, \dots, Q_n$  for  $X$  to a system of relators for  $Y$ . On each step a finite presentation is transformed to a finite presentation. Hence, in the end we get a finite system of relators  $R_1, \dots, R_k$  for the generating set  $X$ .  $\square$

Therefore there are finitely many elements  $\alpha_1, \dots, \alpha_n$  of  $\pi_1(\mathcal{F})$  which normally generate the kernel  $\text{Ker}(\phi)$  of

$$\phi : \pi_1(\mathcal{F}) \rightarrow \pi_1(\tilde{X})$$

We shall identify  $\alpha_j$  and the corresponding loops on  $\mathcal{F}$ . Thus there is a closed metric disk  $D$  centered at the origin in  $\mathbb{H}^2 = \tilde{S}$  such that each  $\alpha_j, j = 1, \dots, n$ , is contractible in  $U = \tilde{f}^{-1}(D)$ . This implies that each  $\alpha \in \text{Ker}(\phi)$  is contractible in  $\tilde{f}^{-1}(D)$ . We will assume that the boundary of  $D$  contains no critical values of  $\gamma$ . However we have infinitely many critical values of  $\tilde{f}$  outside of the disk  $D$ . Let  $z$  be one of them and  $D'$  be a closed topological disk in  $\mathbb{H}^2$  which contains both  $D$  and  $z$  and does not contain any critical values of  $\tilde{f}$  which are not in  $\{z\} \cup D$ . Homotopically the Morse surgery corresponding to  $z$  amounts to attaching 2-cells along certain loops  $\alpha \subset \mathcal{F}$ . Thus  $\alpha \in \text{Ker}(\phi)$ , which implies that  $\alpha$  is contractible in  $U$ . It follows that we get an immersed homotopically nontrivial 2-sphere  $\zeta \subset \tilde{f}^{-1}(D')$ . The space  $\tilde{X}$  is obtained from  $\tilde{f}^{-1}(D')$  by attaching only 2-handles, thus the homotopy class  $[\zeta]$  is nontrivial in  $\pi_2(\tilde{X})$  which contradicts asphericity of  $\tilde{X}$ . This concludes the proof in the case when  $\tilde{f}$  is a complex Morse function.

**Remark 6** *J. Kollar had suggested an argument which reduces the general case to the case of holomorphic Morse function provided that no irreducible component of each singular fiber of  $\tilde{f}$  has multiplicity  $> 1$ . Namely, perturb  $\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^2$  in a  $Q$ -equivariant manner to a smooth map  $g : \tilde{X} \rightarrow \mathbb{H}^2$  with connected fibers so that:*

- (a) *The sets of critical values of  $g$  and  $\tilde{f}$  are equal.*
- (b) *If  $s$  is a critical value of  $g$  (and  $\tilde{f}$ ) and  $C_s \subset \mathbb{H}^2$  is a small circle around  $s$  then the 3-manifolds  $g^{-1}(C_s)$ ,  $\tilde{f}^{-1}(C_s)$  are homeomorphic.*
- (c) *The mapping  $g$  is a holomorphic Morse function near each singular fiber.*

*Then apply the same arguments as before to the function  $g$  to conclude that neither  $\tilde{f}$  nor  $g$  has critical points.*

*However, technically it seems (at least to me) easier to apply the direct topological arguments below than to analyze the special case when a singular fiber of  $\tilde{f}$  has an irreducible component of multiplicity  $> 1$ .*

We now consider the general case. We will run essentially the same arguments as in the case of holomorphic Morse function. Let  $\Sigma = \Sigma(\tilde{f})$  denote the set of critical values of the holomorphic function  $\tilde{f}$ ,  $\tilde{S}' := \tilde{S} - \Sigma$  and  $\tilde{X}' := \tilde{f}^{-1}(S')$ .

**Lemma 7** (1) *The fundamental group of a generic fiber  $\mathcal{F}$  of  $\tilde{f}$  maps onto  $K = \pi_1(\tilde{X})$ . (2) No singular fiber of  $\tilde{f}$  is a multicurve, i.e. a singular fiber of  $\tilde{f}$  cannot be a smooth complex curve.*

*Proof.* The restriction  $\tilde{f}'$  of  $\tilde{f}$  to  $\tilde{X}'$  is a (nonsingular) fibration with connected fibers, thus  $\pi_1(\mathcal{F})$  is the kernel of the homomorphism

$$\pi_1(\tilde{f}') : \pi_1(\tilde{X}') \rightarrow \pi_1(\tilde{S}')$$

In particular, the subgroup  $\pi_1(\mathcal{F})$  is normal in  $\pi_1(\tilde{X}')$ . For each puncture  $s_i \in \Sigma$  choose a small loop on  $\tilde{S}'$  going once around  $s_i$  and choose a homeomorphic lift  $\gamma_i$  of this loop to  $\tilde{X}'$ . Then the group  $\pi_1(\tilde{X}')$  is generated by  $\pi_1(\mathcal{F})$  and by the loops  $\gamma_i, s_i \in \Sigma$ . Let  $D_{s_i}$  denote a small metric disk on  $\mathbb{H}^2$  centered at  $s_i \in \Sigma$  (so that  $D_{s_i} \cap \Sigma = \{s_i\}$ ). If for some  $s_i$  the fundamental group of  $\pi_1(\partial \tilde{f}^{-1}(D_{s_i}))$  does not map

onto  $\pi_1(\tilde{f}^{-1}(D_{s_i}))$  then it is true for infinitely many points  $s \in \Sigma$  (all the points in the  $Q$ -orbit of  $s_i$ ), thus the group  $K$  cannot be finitely generated. Thus the map

$$\pi_1(\tilde{X}') \rightarrow \pi_1(\tilde{X})$$

is onto. Since  $\gamma_i$ -s belong to the kernel of this map we conclude that the group  $\pi_1(\mathcal{F})$  maps onto  $\pi_1(\tilde{X})$ .

If  $\mathcal{F}_{s_i} = \tilde{f}^{-1}(s_i)$ ,  $s_i \in \Sigma$ , is a multicurve then

$$\pi_1(\partial \tilde{f}^{-1}(D_{s_i})) \rightarrow \pi_1(\tilde{f}^{-1}(D_{s_i})) = \pi_1(\mathcal{F}_{s_i})$$

is not onto (Lemma 4), which contradicts our assumptions. This proves the second assertion of Lemma.  $\square$

Now suppose that  $f : X \rightarrow S$  is not a nonsingular holomorphic fibration. Thus the map  $\tilde{f}$  has at least one fiber which is not a smooth complex curve. (By Lemma 7 each singular fiber has to be of this type.) Our goal is to show that this assumption leads to a contradiction. Let  $T \subset \mathbb{H}^2$  be a locally finite embedded tree whose vertex set is  $\Sigma$  (this tree of course is not  $\pi_1(S)$ -invariant). We can assume that edges of  $T$  are geodesics in  $\mathbb{H}^2$ . For each vertex  $s \in \Sigma$  of  $T$  we choose a small closed metric disk  $D_s$  centered at  $s$  such that  $D_s \cap T$  is equal to the intersection of  $D_s$  and open edges of  $T$  emanating from  $s$ . If  $T' \subset T$  is a subtree then  $N(T')$  will denote the union of  $T'$  and disks  $D_s$  for those vertices  $s$  of  $T$  which belong to  $T'$ . Let  $Y(T') := \tilde{f}^{-1}(N(T'))$ .

Since  $\tilde{f}$  is a smooth fibration away from singular fibers it follows that the inclusion

$$Y(T) \hookrightarrow \tilde{X}$$

is a homotopy-equivalence. Therefore we restrict our attention to the topology of  $Y(T)$ .

Let  $T'$  be a finite subtree of  $T$  which is the convex hull of its vertices.

**Lemma 8** *The homomorphism*

$$\pi_2(Y(T')) \rightarrow \pi_2(Y(T))$$

*is injective.*

*Proof.* It is enough to prove this assertion for the lifts of  $Y(T')$ ,  $Y(T)$  to the universal cover of  $X$ . Since  $X$  is aspherical, its universal cover  $\tilde{X}$  cannot contain compact complex curves, hence the lift of  $\tilde{f}^{-1}(t)$ ,  $t \in T - \Sigma$  to  $\tilde{X}$  is a noncompact surface. Therefore this lift has trivial  $H_2$  and the assertion follows from the Meyer-Vietors sequence.  $\square$

Let  $s \in \Sigma - T'$  be a vertex of  $T$  which is connected to  $T'$  by an edge  $[ss']$ ,  $s' \in \Sigma \cap T'$ . Note that the inclusions

$$Y(T') \hookrightarrow Y(T' \cup [s's]), \quad Y(s) \hookrightarrow Y([ss'])$$

are homotopy-equivalences. Here and in what follows  $[ss']$  denotes the half-open edge connecting  $s$  to  $s'$ :  $s \in [ss']$ ,  $s' \notin [ss']$ .

**Lemma 9** Suppose that  $\pi_1(Y(T')) \rightarrow \pi_1(Y(T))$  is a monomorphism. Then  $\pi_2(Y(T' \cup [ss'])) \neq 0$ .

*Proof.* Let  $t \in [ss']$  be the midpoint. Then there is a subsurface  $\mathcal{F}' \subset \mathcal{F}_t$  such that:

- (a) No boundary loop of  $\mathcal{F}'$  is nil-homotopic in  $\mathcal{F}_t$ .
- (b) The image of  $\pi_1(\mathcal{F}')$  in  $\pi_1(Y([ss']))$  is trivial.

The subsurface  $\mathcal{F}'$  appears as follows: let  $p \in \mathcal{F}_s$  be a singular point, then  $\mathcal{F}'$  is a part of  $\mathcal{F}_t$  corresponding to the Milnor fiber in  $S_\epsilon(p)$ , see section 2. If a boundary loop of  $\mathcal{F}'$  is nil-homotopic in  $\mathcal{F}_t$  then  $\tilde{X}$  contains a rational complex curve which contradicts the assumption that  $\pi_2(X) = 0$ .

Therefore, the assumption of Lemma implies that the image of  $\pi_1(\mathcal{F}')$  in  $\pi_1(Y(T' \cup [ss']))$  is trivial. Consider the total lift  $\hat{\mathcal{F}'}$  of  $\mathcal{F}'$  to the universal cover  $\hat{X}$  of  $X$ , then  $\hat{\mathcal{F}'}$  is contained in the lift  $\hat{\mathcal{F}}_t$  of  $\mathcal{F}_t$  to  $\hat{X}$ . Note that no component of  $\hat{\mathcal{F}}_t - \hat{\mathcal{F}'}$  is bounded (otherwise after degeneration of  $\hat{\mathcal{F}}_s$  to a singular fiber we will get a compact complex curve in  $\hat{X}$  which is impossible). If  $\mathcal{F}'$  is not a planar surface then  $\hat{\mathcal{F}'}$  contains a non-separating loop, otherwise a component of  $\partial \hat{\mathcal{F}'}$  is not nil-homologous in  $\hat{\mathcal{F}}_t$ . In the both cases we apply Meyer-Vietors arguments to get a homologically nontrivial spherical cycle in  $\hat{Y}(T' \cup [ss'])$ , thus  $\pi_2(Y(T' \cup [ss'])) \neq 0$ .  $\square$

Since  $K$  is assumed to be finitely-presentable, there are finitely many elements  $\alpha_i \in \pi_1(\mathcal{F})$  which normally generate the kernel of  $\pi_1(\mathcal{F}) \rightarrow \pi_1(Y)$ . Thus there is a finite subtree  $T' \subset T$  such that all the loops  $\alpha_i$  are nil-homotopic in  $Y(T')$ . Since  $\pi_1(Y(T'))$  maps onto  $\pi_1(Y(T))$  (Lemma 7) it follows that  $\pi_1(Y(T')) \rightarrow \pi_1(Y(T))$  is an isomorphism. Hence for an edge  $[ss']$  of  $T$  which has one vertex in  $T'$  and the other vertex in  $T - T'$  we have:

$$\pi_2(Y(T' \cup [ss'])) \neq 0$$

(according to Lemma 9). Now we apply Lemma 8 to conclude that  $\pi_2(Y(T)) \neq 0$ . However  $\pi_2(Y(T)) \cong \pi_2(\tilde{X}) = 0$  since  $X$  is aspherical. This contradiction proves Theorem 2.  $\square$

## 5 Complex-hyperbolic surfaces

Let  $B \subset \mathbb{C}^2$  be the unit ball. We will give  $B$  the *Kobayashi metric*, this metric can be described as follows. Let  $p, q \in B$  be distinct points, there is a unique complex line  $L \subset \mathbb{C}^2$  so that  $p, q \in B \cap L$ . Now identify  $B \cap L$  with the hyperbolic plane  $\mathbb{H}^2$  where the curvature is normalized to be  $-1$ . Finally let  $d(p, q) := d_{\mathbb{H}^2}(p, q)$ . Then the *complex-hyperbolic plane*  $\mathbb{H}_{\mathbb{C}}^2$  is the unit ball  $B$  with the Kobayashi distance  $d$ . It turns out that the Kobayashi distance  $d$  is induced by a Riemannian metric  $\rho$  on  $B$ . Below we list some properties of the complex-hyperbolic plane  $\mathbb{H}_{\mathbb{C}}^2$ , we refer to [8], [2], [5], [20] for detailed discussion.

- (a)  $\rho$  is Kähler.
- (b) The sectional curvature of  $\rho$  is pinched between the constants  $-1$  and  $-1/4$ .

(c) The group of biholomorphic automorphisms of  $B$  equals the identity component in the isometry group of  $\mathbb{H}_{\mathbb{C}}^2$  which is isomorphic to the  $PU(2, 1)$  so that  $B$  is the symmetric space for the group  $PU(2, 1)$ :  $B = PU(2, 1)/K$  where  $K \cong U(2)$  is a maximal compact subgroup in  $PU(2, 1)$ .

(d) Let  $\Gamma$  be a torsion-free uniform lattice in  $PU(2, 1)$ . The quotient  $B/\Gamma$  is a compact Kähler surface which is actually a smooth complex algebraic surface. The quotient  $B/\Gamma$  is called a *complex-hyperbolic surface*.

(e) For each compact complex-hyperbolic surface we have the following identity between the Chern classes:  $c_1^2 = 3c_2$ , i.e.  $\chi = 3\tau$  where  $\chi$  is the Euler characteristic and  $\tau$  is the signature.

(f) If  $X$  is a smooth compact complex algebraic surface for which the equality  $c_1^2 = 3c_2$  holds, then the universal cover of  $X$  is biholomorphic to either  $\mathbb{H}_{\mathbb{C}}^2$ , or  $\mathbb{C}^2$ , or the complex-projective plane  $\mathbb{P}_{\mathbb{C}}^2$ .

The key fact about complex-hyperbolic surfaces which will be used in this paper is the following recent theorem of K. Liu [12]:

**Theorem 10** *Let  $X$  be a compact complex-hyperbolic surface. Then  $X$  does not admit nonsingular holomorphic fibrations over complex curves.*

## 6 Incoherent example

Recall that a group  $\Gamma$  is called *coherent* if every finitely-generated subgroup  $\Gamma' \subset \Gamma$  is also finitely presentable. Examples of coherent groups include free groups, surface groups, 3-manifold groups (see [19]) and certain groups of cohomological dimension 2 (see [7], [15]). The simplest example of noncoherent group is  $\mathbb{F}_2 \times \mathbb{F}_2$ , where  $\mathbb{F}_2$  is the free group on two generators. (The finitely generated infinitely presentable subgroup in  $\mathbb{F}_2 \times \mathbb{F}_2$  is the kernel of the homomorphism  $\phi : \mathbb{F}_2 \times \mathbb{F}_2 \rightarrow \mathbb{Z}$  where  $\phi$  maps each free generator of each  $\mathbb{F}_2$  to the generator of  $\mathbb{Z}$ .) Thus there is a uniform lattice in the Lie group  $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$  which is not coherent. The first example of noncoherent discrete geometrically finite subgroup of  $Isom(\mathbb{H}^4)$  was constructed in [11], [18]. Later on this example was generalized in [4] to a uniform lattice in  $Isom(\mathbb{H}^4)$ .

As an application of the main result of this paper we show that certain uniform lattices in  $PU(2, 1) = Isom(\mathbb{H}_{\mathbb{C}}^2)$  are not coherent (these are the first known examples of incoherent discrete subgroups of  $PU(2, 1)$ ). The groups which we consider were known before (see [13], [2], [5]) however their incoherence was unknown.

**Lemma 11** *Suppose that  $X$  is a compact complex-hyperbolic surface whose fundamental group  $\pi$  fits into a short exact sequence*

$$1 \rightarrow K \rightarrow \pi \rightarrow Q = \pi_1(S) \rightarrow 1$$

*where  $S$  is a compact hyperbolic Riemann surface and the group  $K$  is finitely generated. Then  $K$  is not finitely presentable.*

*Proof.* Suppose that  $K$  is finitely presentable. The surface  $X$  is aspherical since its universal cover is the complex ball. Then by Theorem 2 the projection  $\pi \rightarrow Q$  is induced by a nonsingular holomorphic fibration of the surface  $X$ . On the other hand, complex-hyperbolic surfaces do not admit such fibrations by [12].  $\square$

Now we describe an example the fundamental group of a complex-hyperbolic surface satisfying the conditions of Lemma 11 following [13]. Define automorphisms  $\phi, \psi$  of the free group on three generators  $A_1, A_2, A_3$  by

$$\phi(A_1) = A_1 A_2 A_1^{-1}, \phi(A_2) = A_1 A_3 A_1^{-1}, \phi(A_3) = A_1$$

$$\psi(A_1) = (A_1 A_2) A_3 (A_1 A_2)^{-1}, \psi(A_2) = A_1 A_2 A_1^{-1}, \psi(A_3) = A_1$$

Ron Livne [13] constructed a uniform lattice  $\Gamma_{d,N}$  in  $PU(2, 1)$  with the presentation

$$\langle x, y, A_1, A_2, A_3 \mid xA_i x^{-1} = \phi(A_i), yA_i y^{-1} = \psi(A_i) \quad (1 \leq i \leq 3),$$

$$x^3 = y^2 = A_1 A_2 A_3, (A_1 A_2 A_3)^{2d} = A_1^2 = A_2^2 = A_3^2 = (yx^{-1})^N = 1 \rangle$$

where  $(N, d) \in \{(7, 7), (8, 4), (9, 3), (12, 2)\}$ . Note that the subgroup  $K_d$  generated by  $A_1, A_2, A_3$  in  $\Gamma_{d,N}$  is normal and finitely generated, the quotient  $\Gamma_{d,N}/K_d$  is the hyperbolic triangle group

$$\Delta_N := \langle x, y \mid x^3 = y^2 = (yx^{-1})^N = 1 \rangle$$

since  $N \geq 7$ . Now fix a pair  $(N, d)$  from the above list and let  $\Gamma := \Gamma_{d,N}, \Delta := \Delta_N, K := K_d$ . Let  $\Delta' < \Delta$  be a torsion-free subgroup of finite index and  $\Gamma' < \Gamma$  be the pull-back of  $\Delta'$  to  $\Gamma$ . Then  $\Gamma'$  fits into short exact sequence

$$1 \rightarrow K \rightarrow \Gamma' \rightarrow \Delta' \rightarrow 1$$

The group  $\Gamma'$  still has torsion, so let  $\pi$  be a torsion-free subgroup of  $\Gamma'$ ,  $K' := \pi \cap K, Q := \pi/K'$ . Clearly  $K'$  is finitely generated and  $Q$  is the fundamental group of a compact hyperbolic Riemann surface. The group  $\pi$  acts freely discretely cocompactly on  $\mathbb{H}_{\mathbb{C}}^2$  and hence is the fundamental group of the compact complex-hyperbolic surface  $X = \mathbb{H}_{\mathbb{C}}^2/\pi$ . By Lemma 11 the group  $K'$  is not finitely presentable.

**Remark 12** Bill Goldman had told me long ago about Livne's example as a candidate for non-coherence, however until recently I did not know how to prove that the group  $K$  is not finitely presentable.

Note that the group  $K$  is not geometrically finite and its limit set is the whole sphere at infinity of  $\mathbb{H}_{\mathbb{C}}^2$  (since  $K$  is normal in  $\Gamma$ ).

**Question 13** Let  $\Gamma \subset PU(2, 1)$  be a finitely generated discrete subgroup whose limit set is not the whole sphere at infinity of  $\mathbb{H}_{\mathbb{C}}^2$ . Is  $\Gamma$  finitely-presentable? Is  $\Gamma$  geometrically finite?

**Remark 14** *There are several reasons why it is difficult to construct finitely generated geometrically infinite subgroups of  $PU(2, 1)$ . One of them is the following result due to M. Ramachandran:*

*Let  $\Gamma$  be a discrete subgroup of  $PU(2, 1)$  which does not contain parabolic elements and which acts cocompactly on a component  $\Omega_0$  of the domain of discontinuity  $\Omega(\Gamma) \subset \partial_\infty \mathbb{H}^2_{\mathbb{C}}$ . Then  $\Gamma$  is geometrically finite and  $\Omega_0 = \Omega(\Gamma)$ .*

*(Instead of assuming that  $\Gamma$  contains no parabolic elements it is enough to assume that each maximal parabolic subgroup of  $\Gamma$  is isomorphic to a lattice in the 3-dimensional Heisenberg group. )*

**Question 15** *Is there a compact real-hyperbolic 4-manifold  $X$  whose fundamental group fits into a short exact sequence:*

$$1 \rightarrow K \rightarrow \pi_1(X) \rightarrow Q \rightarrow 1$$

*where  $K$  is finitely presentable or even a surface group and  $Q$  is a hyperbolic surface group ?*

More generally:

**Question 16** *Is there a Gromov-hyperbolic group  $\pi$  which fits into a short exact sequence:*

$$1 \rightarrow K \rightarrow \pi \rightarrow Q \rightarrow 1$$

*where  $K$  and  $Q$  are closed hyperbolic surface groups?*

Note that Lee Mosher [17] constructed similar example when  $K$  is a closed hyperbolic surface group and  $Q$  is a free nonabelian group.

**Question 17** *Let  $\Gamma_g$  be the mapping class group of a compact surface of genus  $g$ . Is there  $g$  and a finitely generated non-free subgroup  $Q$  of  $\Gamma_g$  which consists only of the identity and pseudo-Anosov elements?*

Mosher's example comes from a "Schottky-type" subgroup  $Q$  in  $\Gamma_g$  where  $K$  is the fundamental group of a genus  $g$  surface.

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